Statistical approach to thermohaline turbulent convection

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Thermohaline turbulence
DI formalism
Perturbation analysis

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Abstract

This paper presents a statistical analysis of the turbulent motion of a viscous incompressible fluid subject to the thermohaline convection. The analysis is in the spirit of the works of Yoshizawa who introduced a new perturbation method for solving hydrodynamical equations. The results obtained in this paper generalize the formulas of the above-mentioned author and enable us to obtain general expressions for the velocity, velocity-temperature, and velocity-salinity covariances. In particular, the Reynolds stress, temperature flux, and salinity flux are evaluated in the inertial-range turbulence.

1. Introduction

Let D denote a confined region of space which is occupied by an incompressible fluid of density ρ and kinematic viscosity ν. The motion of this fluid will be described by the velocity vector field $w(x,t)=[w^1(x,t), w^2(x,t), w^3(x,t)]$, $x=(x^1, x^2, x^3)\in D$.

It is well known that if a physical system consisting of a viscous fluid and rigid bodies is not subjected to any external action, it will tend to a state of rest.

We now submit the fluid to a steady action measured by parameter τ (Reynolds number, Rayleigh number, Grashof number, *etc*). When $\tau=0$, the fluid is at rest. For $\tau>0$ we first obtain a steady state, *ie* the physical parameters describing the fluid at any point (velocity, temperature, salinity, *etc*) are constant in time, but the fluid is no longer in equilibrium. This situation prevails for small values of τ. For sufficiently large τ, the fluid motion becomes very complicated, irregular, and chaotic, and we have turbulence [7].

In nature the turbulent motion (turbulent convection, turbulent thermohaline convection) is important in many areas of geophysics, astrophysics, and oceanology. Typical examples include atmospheric convection, stellar convection, and picnocline erosion in the ocean [1, 6].

In this paper we consider the turbulent motion of a viscous incompressible fluid subject to the thermohaline convection. A strict, mathematical approach to the turbulent thermohaline convection was given by Icha [2], but the treatment remains purely formal at this stage. In our analysis, we extend the Yoshizawa's recent work based on DI formalism for thermal convection [11] and find general expressions for the velocity covariance and the velocity-temperature and velocity-salinity covariances. The results obtained are applied to evaluate the Reynolds stress, temperature flux, and the salinity flux in the inertial range. Althoügh made in the spirit of the works of Yoshizawa [10, 11], the present analysis is a little more complicated. The reader who is interested in the details of the manipulation should consult Yoshizawa a works [10, 11] and Leslie's book [5].

2. Basic equations

In the Oberbeck-Boussinesq approximation the continuity equation, the Navier--Stokes equations, and the diffusion equations for temperature and salinity have the following general form, as in [3]:

$$\frac{\partial v^\alpha}{\partial x^\alpha}=0, \tag{2.1}$$

$$\frac{\partial v^\alpha}{\partial t}+\frac{\partial v^\alpha v^\beta}{\partial x^\beta}=-\frac{\partial \pi}{\partial x^\alpha}+\nu\frac{\partial^2 v^\alpha}{\partial x^\beta \partial x^\beta}+A^\alpha \psi+B^\alpha s, \tag{2.2}$$

$$\frac{\partial \psi}{\partial t}+\frac{\partial v^\beta \psi}{\partial x^\beta}=k_\psi \frac{\partial^2 \psi}{\partial x^\beta \partial x^\beta}, \tag{2.3}$$

$$\frac{\partial s}{\partial t}+\frac{\partial v^\beta s}{\partial x^\beta}=k_s \frac{\partial^2 s}{\partial x^\beta \partial x^\beta}, \tag{2.4}$$

where π is the pressure divided by the density, $A^\alpha \psi+B^\alpha S$ is a destabilizing force per unit mass, k_ψ and k_s are the coefficients of thermal and solutal diffusion, respectively. We employ the summation convention over repeated indices.

In the case of the thermohaline convection, the destabilizing force is due to the acceleration of gravity $\vec{g}=(0,0,-g)$ acting through thermal and solutal expansion, so that:

$$A^\alpha=g^\alpha \alpha_0=-g^\alpha\left[\frac{1}{\rho}\left(\frac{\partial \rho}{\partial \psi}\right)_{s,p}\right],$$

$$B^\alpha=g^\alpha \gamma_0=g^\alpha\left[\frac{1}{\rho}\left(\frac{\partial \rho}{\partial s}\right)_{\psi,p}\right].$$

3. Statistical formulation

We begin by introducing two space and time scales. Assuming that the space and time variation of mean fields is slow, as compared with that of the fluctuating fields, we write [10, 11]:

$$x^\alpha,\ X^\alpha(=\lambda x^\alpha),\ t,\ T(=\lambda t). \tag{3.1}$$

Here λ is an ordering parameter and will be assumed to equal a unity at the end of the calculation. Then $\vec{v}$, π, ψ and S are written as:

$$\vec{v}=\vec{U}(\vec{X},T)+\vec{u}(\vec{x},\vec{X},t,T), \qquad \langle\vec{u}\rangle=0, \tag{3.2}$$

$$\pi=P(\vec{X},T)+p(\vec{x},\vec{X},t,T), \qquad \langle p\rangle=0, \tag{3.3}$$

$$\psi=\Theta(\vec{X},T)+\vartheta(\vec{x},\vec{X},t,T), \qquad \langle\vartheta\rangle=0, \tag{3.4}$$

$$s=\Gamma(\vec{X},T)+\gamma(\vec{x},\vec{X},t,T), \qquad \langle\gamma\rangle=0, \tag{3.5}$$

where $\langle\ \rangle$ denotes the mean over an ensemble. By virtue of (3.2)–(3.5), (2.1)–2.4) may be rewritten, up to $O(\lambda)$, as:

$$\frac{\partial u^\beta}{\partial x^\beta}+\lambda\frac{\partial u^\beta}{\partial X_\beta}=0, \tag{3.6}$$

$$\begin{aligned}\frac{\partial u^\alpha}{\partial t}+U^\beta\frac{\partial u^\alpha}{\partial x^\beta}+\frac{\partial u^\alpha u^\beta}{\partial x^\beta}+\frac{\partial p}{\partial x^\alpha}-\nu\frac{\partial^2 u^\alpha}{\partial x^\beta\partial x^\beta}=\lambda\Big(&-u^\beta\frac{\partial U^\alpha}{\partial X^\beta}+A^\alpha\lambda^{-1}\vartheta+\\ &+B^\alpha\lambda^{-1}\gamma+A^\alpha\lambda^{-1}\Theta+B^\alpha\lambda^{-1}\Gamma-\frac{\partial U^\alpha}{\partial T}-U^\beta\frac{\partial U^\alpha}{\partial X^\beta}-\frac{\partial P}{\partial X^\alpha}-\frac{\partial u^\alpha}{\partial T}+\\ &-U^\beta\frac{\partial u^\alpha}{\partial X^\beta}-\frac{\partial u^\alpha u^\beta}{\partial X^\beta}-\frac{\partial p}{\partial X^\alpha}+2\nu\frac{\partial^2 u^\alpha}{\partial x^\beta\partial X^\beta}\Big),\end{aligned} \tag{3.7}$$

$$\begin{aligned}\frac{\partial\vartheta}{\partial t}+U^\beta\frac{\partial\vartheta}{\partial x^\beta}+\frac{\partial u^\beta\vartheta}{\partial x^\beta}-k_\psi\frac{\partial^2\vartheta}{\partial x^\beta\partial x^\beta}=\lambda\Big(&-u^\beta\frac{\partial\Theta}{\partial X^\beta}-\frac{\partial\Theta}{\partial T}-U^\beta\frac{\partial\Theta}{\partial X^\beta}+\\ &-\frac{\partial\vartheta}{\partial T}-U^\beta\frac{\partial\vartheta}{\partial X^\beta}-\frac{\partial u^\beta\vartheta}{\partial X^\beta}+2k_\psi\frac{\partial^2\vartheta}{\partial x^\beta\partial X^\beta}\Big),\end{aligned} \tag{3.8}$$

$$\begin{aligned}\frac{\partial\gamma}{\partial t}+U^\beta\frac{\partial\gamma}{\partial x^\beta}+\frac{\partial u^\beta\gamma}{\partial x^\beta}-k_s\frac{\partial^2\gamma}{\partial x^\beta\partial x^\beta}=\lambda\Big(&-u^\beta\frac{\partial\Gamma}{\partial X^\beta}-\frac{\partial\Gamma}{\partial T}-U^\beta\frac{\partial\Gamma}{\partial X^\beta}+\\ &-\frac{\partial\gamma}{\partial T}-U^\beta\frac{\partial\gamma}{\partial X^\beta}-\frac{\partial u^\beta\gamma}{\partial X^\beta}+2k_s\frac{\partial^2\gamma}{\partial x^\beta\partial X^\beta}\Big).\end{aligned} \tag{3.9}$$

We introduce a Galilean transformation such that

$$\xi^\alpha=x^\alpha-U^\alpha t \tag{3.10}$$

to remove the convection effect due to the mean flow $\vec{U}(\vec{X},T)$ from (3.7)–(3.9), and the Fourier transformation defined by

$$f(\vec{k},\vec{X},t,T)=\frac{1}{(2\pi)^3}\int_{\vec{\xi}}f(\vec{\xi},\vec{X},t,T)e^{i\vec{k}\cdot\vec{\xi}}, \tag{3.11}$$

where a special symbol $\int_{\vec{\xi}}=\iiint_{-\infty}^{\infty}d\xi^1d\xi^2d\xi^3$ is introduced.

Application of (3.10) and (3.11) to (3.6)–(3.9), after the elimination of the pressure with the aid of the continuity equation, gives:

$$\frac{\partial u^\alpha(\vec{k},t)}{\partial t}+\nu k^2 u^\alpha(\vec{k},t)-M^{\alpha\beta\gamma}(\vec{k})\int\limits_{\vec{p}}\int\limits_{\vec{q}}\delta(\vec{k}-\vec{p}-\vec{q})\,u^\beta(\vec{p},t)\,u^\gamma(\vec{q},t)=$$

$$=\lambda\Bigg[-D^{\alpha\beta}(\vec{k})\,u^\gamma(\vec{k},t)\frac{\partial U^\beta}{\partial X^\gamma}+D^{\alpha\beta}(\vec{k})\,A^\beta\lambda^{-1}\vartheta(\vec{k},t)+D^{\alpha\beta}(\vec{k})\,B^\beta\lambda^{-1}\gamma(\vec{k},t)+$$

$$-\delta(\vec{k})\left(\frac{\partial U^\alpha}{\partial T}+U^\beta\frac{\partial U^\alpha}{\partial X^\beta}+\frac{\partial p}{\partial X^\alpha}-A^\alpha\lambda^{-1}\Theta-B^\alpha\lambda^{-1}\Gamma\right)-\frac{\partial u^\alpha(\vec{k},t)}{\partial T}-U^\beta\frac{\partial u^\alpha(\vec{k},t)}{\partial X^\beta}$$

$$+N^{\alpha\beta\gamma\delta}(\vec{k})\int\limits_{\vec{p}}\int\limits_{\vec{q}}\delta(\vec{k}-\vec{p}-\vec{q})\frac{\partial u^\gamma(\vec{p},t)\,u^\delta(\vec{q},t)}{\partial X^\beta}-2\nu i k^\beta\frac{\partial u^\alpha(\vec{k},t)}{\partial X^\beta}\Bigg], \tag{3.12}$$

$$\frac{\partial\vartheta(\vec{k},t)}{\partial t}+k_\psi k^2\vartheta(\vec{k},t)-ik^\alpha\int\limits_{\vec{p}}\int\limits_{\vec{q}}\delta(\vec{k}-\vec{p}-\vec{q})\,u^\alpha(\vec{p},t)\,\vartheta(\vec{q},t)=$$

$$=\lambda\Bigg[-u^\alpha(\vec{k},t)\frac{\partial\Theta}{\partial X^\alpha}-\delta(\vec{k})\left(\frac{\partial\Theta}{\partial T}+U^\alpha\frac{\partial\Theta}{\partial X^\alpha}\right)-\frac{\partial\vartheta(\vec{k},t)}{\partial T}-U^\alpha\frac{\partial\vartheta(\vec{k},t)}{\partial X^\alpha}+$$

$$-\int\limits_{\vec{p}}\int\limits_{\vec{q}}\delta(\vec{k}-\vec{p}-\vec{q})\frac{\partial u^\alpha(\vec{p},t)\,\vartheta(\vec{q},t)}{\partial X^\alpha}-2k_\psi ik^\alpha\frac{\partial\vartheta(\vec{k},t)}{\partial X^\alpha}\Bigg] \tag{3.13}$$

and

$$\frac{\partial\gamma(\vec{k},t)}{\partial t}+k_s k^2\gamma(\vec{k},t)-ik^\alpha\int\limits_{\vec{p}}\int\limits_{\vec{q}}\delta(\vec{k}-\vec{p}-\vec{q})\,u^\alpha(\vec{p},t)\,\gamma(\vec{q},t)=$$

$$=\lambda\Bigg[-u^\alpha(\vec{k},t)\frac{\partial\Gamma}{\partial X^\alpha}-\delta(\vec{k})\left(\frac{\partial\Gamma}{\partial T}+U^\alpha\frac{\partial\Gamma}{\partial X^\alpha}\right)-\frac{\partial\gamma(\vec{k},t)}{\partial T}-U^\alpha\frac{\partial\gamma(\vec{k},t)}{\partial X^\alpha}+$$

$$-\int\limits_{\vec{p}}\int\limits_{\vec{q}}\delta(\vec{k}-\vec{p}-\vec{q})\frac{\partial u^\alpha(\vec{p},t)\,\gamma(\vec{q},t)}{\partial X^\alpha}-2k_s ik^\alpha\frac{\partial\gamma(\vec{k},t)}{\partial X^\alpha}\Bigg], \tag{3.14}$$

where $\delta(\cdot)$ is the Dirac delta function $D^{\alpha\beta}(\vec{k})$ is the transverse projector $D^{\alpha\beta}(\vec{k})=\delta^{\alpha\beta}-k^\alpha k^\beta/k^2$), the inertial-transfer operator $M^{\alpha\beta\gamma}(\vec{k})$ is defined by

$$M^{\alpha\beta\gamma}(\vec{k})=\frac{1}{2}\left[k^\beta D^{\alpha\gamma}(\vec{k})+k^\gamma D^{\alpha\beta}(\vec{k})\right] \text{ and } N^{\alpha\beta\gamma\delta}(\vec{k})=\frac{2k^\gamma}{k^2}M^{\beta\alpha\delta}(\vec{k})-D^{\alpha\delta}(\vec{k})\,\delta^{\beta\gamma}.$$

We should note that ϑ and γ are of $O(\lambda)$ owing to its linearity in (3.13) and (3.14).

We next expand u^α, ϑ and γ in the power series of λ [10, 11]:

$$u^\alpha(\vec{k}, t) = u_0^\alpha(\vec{k}, t) + \lambda u_1^\alpha(\vec{k}, t) + \dots, \tag{3.15}$$

$$\vartheta(\vec{k}, t) = \lambda \vartheta_1(\vec{k}, t) + \dots, \tag{3.16}$$

$$\gamma(\vec{k}, t) = \lambda \gamma_1(\vec{k}, t) + \dots . \tag{3.17}$$

After the manner of Yoshizawa's paper [10, 11] u_1^α, ϑ_1 and γ_1 can be integrated, by using u_0^α, as:

$$\begin{aligned} u_1^\alpha(\vec{k}, t) = &-D^{\beta\gamma}(\vec{k}) \frac{\partial U^\gamma}{\partial X^\delta} \int_{-\infty}^{t} dt_1\, \hat{G}^{\alpha\beta}(\vec{k}, t, t_1) u_0^\delta(\vec{k}, t_1) + \\ &-D^{\beta\gamma}(\vec{k}) A^\gamma \frac{\partial \Theta}{\partial X^\delta} \int_{-\infty}^{t} dt_1\, \hat{G}^{\alpha\beta}(\vec{k}, t, t_1) \int_{-\infty}^{t_1} dt_2\, \hat{G}_\vartheta(\vec{k}, t_1, t_2) u_0^\delta(\vec{k}, t_2) + \\ &-D^{\beta\gamma}(\vec{k}) B^\gamma \frac{\partial \Gamma}{\partial X^\delta} \int_{-\infty}^{t} dt_1\, \hat{G}^{\alpha\beta}(\vec{k}, t, t_1) \int_{-\infty}^{t_1} dt_2\, \hat{G}_\gamma(\vec{k}, t_1, t_2) u_0^\delta(\vec{k}, t_2), \end{aligned} \tag{3.18}$$

$$\vartheta_1(\vec{k}, t) = -\frac{\partial \Theta}{\partial X^\alpha} \int_{-\infty}^{t} dt_1\, \hat{G}_\vartheta(\vec{k}, t, t_1) u_0^\alpha(\vec{k}, t_1), \tag{3.19}$$

$$\gamma_1(\vec{k}, t) = -\frac{\partial \Gamma}{\partial X^\alpha} \int_{-\infty} dt_1\, \hat{G}_\gamma(\vec{k}, t, t_1) u_0^\alpha(\vec{k}, t_1), \tag{3.20}$$

where $\hat{G}^{\alpha\beta}$, $\hat{G}_\vartheta$ an($\hat{G}_\gamma$ are the Green's functions satisfying

$$\begin{aligned} &\frac{\partial \hat{G}^{\alpha\beta}(\vec{k}, t, t')}{\partial t} + \nu k^2 \hat{G}^{\alpha\beta}(\vec{k}, t, t') - iM^{\alpha\gamma\delta}(k) \int_{\vec{p}} \int_{\vec{q}} \delta(\vec{k} - \vec{p} - \vec{q}) u_0^\gamma(\vec{p}, t) \hat{G}^{\delta\beta}(\vec{q}, t, t') = \\ &= \delta^{\alpha\beta} \delta(t - t'), \end{aligned} \tag{3.21}$$

$$\frac{\partial \hat{G}_\vartheta(\vec{k}, t, t')}{\partial t} + k_\psi k^2 \hat{G}_\vartheta(\vec{k}, t, t') - ik^\alpha \int_{\vec{p}} \int_{\vec{q}} \delta(\vec{k} - \vec{p} - \vec{q}) u_0^\alpha(\vec{p}, t) \hat{G}_\vartheta(\vec{q}, t, t') = \delta(t - t'), \tag{3.22}$$

$$\frac{\partial \hat{G}_\gamma(k, t, t')}{\partial t} + k_s k^2 \hat{G}_\gamma(k, t, t') - ik^\alpha \int_{\vec{p}} \int_{\vec{q}} \delta(\vec{k} - \vec{p} - \vec{q}) u_0^\alpha(\vec{p}, t) \hat{G}_\gamma(\vec{q}, t, t') = \delta(t - t') \tag{3.23}$$

respectively.

Let us now introduce the velocity covariance $Q^{\alpha\beta}$, the velocity-temperature covariance Λ, the velocity-salinity covariance Ω, and the averaged Green's functions

$G^{\alpha\beta}$, G_ϑ and G_γ defined by [10, 11]:

$$\langle u^\alpha(\vec{k},t)u^\beta(\vec{k},t')\rangle = Q^{\alpha\beta}(\vec{k},t,t')\delta(\vec{k}+\vec{k}') = D^{\alpha\beta}(\vec{k})Q(k,t,t')\delta(\vec{k}+\vec{k}')+ \\ +\lambda R^{\alpha\beta}(\vec{k},t,t')\delta(\vec{k}+\vec{k}')+\dots, \tag{3.24}$$

$$\langle u^\alpha(\vec{k},t)\vartheta(\vec{k}',t')\rangle = \Lambda^\alpha(\vec{k},t,t')\delta(\vec{k}+\vec{k}') = \lambda R^\alpha(\vec{k},t,t')\delta(\vec{k}+\vec{k}')+\dots, \tag{3.25}$$

$$\langle u^\alpha(\vec{k},t)\gamma(\vec{k}',t')\rangle = \Omega^\alpha(\vec{k},t,t')\delta(\vec{k}+\vec{k}') = \lambda H^\alpha(\vec{k},t,t')\delta(\vec{k}+\vec{k}')+\dots, \tag{3.26}$$

$$\langle \hat{G}^{\alpha\beta}(\vec{k},t,t')\rangle = G^{\alpha\beta}(\vec{k},t,t') = \delta^{\alpha\beta}G(k,t,t'), \tag{3.27}$$

$$\langle \hat{G}_\vartheta(\vec{k},t,t')\rangle = G_\vartheta(k,t,t'), \tag{3.28}$$

$$\langle \hat{G}_\gamma(\vec{k},t,t')\rangle = G_\gamma(k,t,t'). \tag{3.29}$$

We can evaluate these quantities by means of the Direct-Interaction Approximation (DIA) (propagator renormalization in the terminology of quantum-field theory) [4, 5, 10]. The result is that:

$$R^{\alpha\beta}(\vec{k},t,t) = -D^{\alpha\gamma}(\vec{k})D^{\beta\delta}(\vec{k})\left(\frac{\partial U^\gamma}{\partial X^\delta}+\frac{\partial U^\delta}{\partial X^\gamma}\right)\int_{-\infty}^{t} dt_1\, G(k,t,t_1)Q(k,t,t_1)+ \\ -D^{\alpha\gamma}(\vec{k})D^{\beta\delta}(\vec{k})\left(A^\gamma\frac{\partial\Theta}{\partial X^\delta}+A^\delta\frac{\partial\Theta}{\partial X^\gamma}\right)\int_{-\infty}^{t} dt_1\, G(k,t,t_1)\int_{-\infty}^{t_1} dt_2\, G_\vartheta(k,t_1,t_2)Q(k,t,t_2)+ \\ -D^{\alpha\gamma}(\vec{k})D^{\beta\delta}(\vec{k})\left(B^\gamma\frac{\partial\Gamma}{\partial X^\delta}+B^\delta\frac{\partial\Gamma}{\partial X^\gamma}\right)\int_{-\infty}^{t} dt_1\, G(k,t,t_1)\int_{-\infty}^{t_1} dt_2\, G_\gamma(k,t_1,t_2)Q(k,t,t_2), \tag{3.30}$$

$$R^\alpha(\vec{k},t,t) = -D^{\alpha\beta}(\vec{k})\frac{\partial\Theta}{\partial X^\beta}\int_{-\infty}^{t} dt_1\, G_\vartheta(k,t,t_1)Q(k,t,t_1), \tag{3.31}$$

$$H^\alpha(\vec{k},t,t) = -D^{\alpha\beta}(\vec{k})\frac{\partial\Gamma}{\partial X^\beta}\int_{-\infty}^{t} dt_1\, G_\gamma(k,t,t_1)Q(k,t,t_1). \tag{3.32}$$

The Fourier transform (3.24), (3.25), and (3.26) with (3.30), (3.31), and (3.32) gives the Reynold's stress $\langle u^\alpha u^\beta\rangle$, temperature flux $\langle u^\alpha\vartheta\rangle$, and salinity flux $\langle u^\alpha v\rangle$, respectively, as:

$$-\langle u^\alpha u^\beta\rangle = -\chi\delta^{\alpha\beta}+\nu_e^{\alpha\beta\gamma\delta}\left(\frac{\partial U^\delta}{\partial x^\gamma}+\frac{\partial U^\gamma}{\partial x^\delta}\right)+\lambda^{\alpha\beta\gamma\delta}\left(A^\gamma\frac{\partial\Theta}{\partial x^\delta}+A^\delta\frac{\partial\Theta}{\partial x^\gamma}\right)+ \\ +\mu^{\alpha\beta\gamma\delta}\left(B^\gamma\frac{\partial\Gamma}{\partial x^\delta}+B^\delta\frac{\partial\Gamma}{\partial x^\gamma}\right), \tag{3.33}$$

$$-\langle u^\alpha\vartheta\rangle = K_\psi\frac{\partial\Theta}{\partial x^\alpha}, \tag{3.34}$$

and

$$-\langle u^\alpha \gamma \rangle = K_s \frac{\partial \Gamma}{\partial x^\alpha}, \tag{3.35}$$

where we have assumed $\lambda=1$. Here χ, $\nu_e^{\alpha\beta\gamma\delta}$, $\lambda^{\alpha\beta\gamma\delta}$, $\mu^{\alpha\beta\gamma\delta}$, K_ψ and K_s are given by:

$$\chi = \frac{8\pi}{3} \int_{K_M}^{\infty} k^2 Q(k, t, t)\, dk, \tag{3.36}$$

$$\nu_e^{\alpha\beta\gamma\delta} = \begin{cases} \frac{8\pi}{5} \delta^{\alpha\gamma}\delta^{\beta\delta} F_1 & \text{for} \quad \alpha \neq \beta \\ \frac{32\pi}{15} \delta^{(\alpha)\gamma}\delta^{(\alpha)\delta} F_1 & \text{for} \quad \alpha = \beta, \end{cases} \tag{3.37}$$

$$\lambda^{\alpha\beta\gamma\delta} = \begin{cases} \frac{8\pi}{5} \delta^{\alpha\gamma}\delta^{\beta\delta} F_2 & \text{for} \quad \alpha \neq \beta \\ \frac{32\pi}{15} \delta^{(\alpha)\gamma}\delta^{(\alpha)\delta} F_2 & \text{for} \quad \alpha = \beta, \end{cases} \tag{3.38}$$

$$\mu^{\alpha\beta\gamma\delta} = \begin{cases} \frac{8\pi}{5} \delta^{\alpha\gamma}\delta^{\beta\delta} F_3 & \text{for} \quad \alpha \neq \beta \\ \frac{32\pi}{15} \delta^{(\alpha)\gamma}\delta^{(\alpha)\delta} F_3 & \text{for} \quad \alpha = \beta, \end{cases} \tag{3.39}$$

$$K_\psi = \frac{8\pi}{3} \int_{K_M}^{\infty} k^2 dk \int_{-\infty}^{t} dt_1\, G_\vartheta(k, t, t_1) Q(k, t, t_1), \tag{3.40}$$

$$K_s = \frac{8\pi}{3} \int_{K_M}^{\infty} k^2 dk \int_{-\infty}^{t} dt_1\, G_\gamma(k, t, t_1) Q(k, t, t_1), \tag{3.41}$$

where:

$$F_1 = \int_{K_M}^{\infty} k^2 dk \int_{-\infty}^{t} dt_1\, G(k, t, t_1) Q(k, t, t_1), \tag{3.42}$$

$$F_2 = \int_{K_M}^{\infty} k^2 dk \int_{-\infty}^{t} dt_1\, G(k, t, t_1) \int_{-\infty}^{t_1} dt_2\, G_\vartheta(k, t_1, t_2) Q(k, t, t_2), \tag{3.43}$$

$$F_3 = \int_{K_M}^{\infty} k^2 dk \int_{-\infty}^{t} dt_1\, G(k, t, t_1) \int_{-\infty}^{t_1} dt_2\, G_\gamma(k, t_1, t_2) Q(k, t, t_2). \tag{3.44}$$

In (3.36)–(3.44), K_M is the wave number magnitude corresponding to largest eddies or waves in the region considered [10, 11].

4. Reynolds stress, temperature flux, and salinity flux in the inertial-range turbulence

Let us consider turbulent flow at a very high Reynolds number. In such flow molecular viscosity plays no important role, except through the dissipation process of turbulent energy, and the coefficients of thermal and solutal diffusion are negligible. Therefore, the Reynolds stress and the scalar flux may be evaluated by using the inertial-range theory of Yoshizawa [8, 9]. He has shown that the inertial-range turbulence is described by [8–11]:

$$Q(k,t,t')=\sigma(k)\exp[-\omega(k)(t-t')], \tag{4.1}$$

$$G(k,t,t')=\exp[-\omega(k)(t-t')]H(t-t'), \tag{4.2}$$

$$G_\vartheta(k,t,t')=\exp[-\omega_\vartheta(k)(t-t')]H(t-t'), \tag{4.3}$$

where $H(t)$ is the step function, and $\sigma(k)$, $\omega(k)$ and $\omega_\vartheta(k)$ are given by:

$$\sigma(k)=0.118\varepsilon^{2/3}k^{-11/3}, \tag{4.4}$$

$$\omega(k)=0.42\varepsilon^{1/3}k^{2/3}, \tag{4.5}$$

$$\omega_\vartheta(k)=0.67\varepsilon^{1/3}k^{2/3}. \tag{4.6}$$

Here ε is the rate of energy dissipation.

It is easy to prove that (cf [9]):

$$G_\gamma(k,t,t')=\exp[-\omega_\gamma(k)(t-t')]H(t-t') \tag{4.7}$$

and

$$\omega_\gamma(k)=0.67\varepsilon^{1/3}k^{2/3}. \tag{4.8}$$

We substitute (4.1)–(4.8) in (3.36)–(3.44) and find, after integration, that:

$$\chi=0.44\varepsilon^{2/3}L_M^{2/3}, \tag{4.9}$$

$$\nu_e^{\alpha\beta\gamma\delta}=\begin{cases}0.046\varepsilon^{1/3}L_M^{4/3}\delta^{\alpha\gamma}\delta^{\beta\delta} & \text{for} \quad \alpha\neq\beta\\ 0.061\varepsilon^{1/3}L_M^{4/3}\delta^{(\alpha)\gamma}\delta^{(\alpha)\delta} & \text{for} \quad \alpha=\beta,\end{cases} \tag{4.10}$$

$$\lambda^{\alpha\beta\gamma\delta}=\begin{cases}0.0085L_M^2\,\delta^{\alpha\gamma}\delta^{\beta\delta} & \text{for} \quad \alpha\neq\beta\\ 0.011L_M^2\,\delta^{(\alpha)\gamma}\delta^{(\alpha)\delta} & \text{for} \quad \alpha=\beta,\end{cases} \tag{4.11}$$

$$\mu^{\alpha\beta\gamma\delta}=\begin{cases}0.0085L_M^2\,\delta^{\alpha\gamma}\delta^{\beta\delta} & \text{for} \quad \alpha\neq\beta\\ 0.011L_M^2\,\delta^{(\alpha)\gamma}\delta^{(\alpha)\delta} & \text{for} \quad \alpha=\beta,\end{cases} \tag{4.12}$$

$$K_\psi=0.058\varepsilon^{1/3}L_M^{4/3}, \tag{4.13}$$

$$K_s=0.058\varepsilon^{1/3}L_M^{4/3}. \tag{4.14}$$

Note that formulas (4.11), (4.12), and expressions (4.13) and (4.14) are identical in the case of high-Reynolds-number thermohaline turbulence. Thus, formulas (3.33), (3.34), and (3.35) with (4.9)–(4.14) constitute general expressions for the Reynolds stress, temperature flux, and salinity flux in the inertial-range turbulence.

5. Concluding remarks

So far, the theoretical framework of Yoshizawa's theory is neither well substantiated nor understood. Many questions remain unanswered: about convergence, uniqueness of solutions, and the importance of terms, which have been neglected. The present analysis is more complicated, which is not surprising; theories based on DI formalism lead to a set of coupled integral equations which are of complex structure. The aim of this paper has only been to generalize the considerations of the above-mentioned author and apply his formalism to the description of thermohaline turbulence in the case of Oberbeck-Boussinesq approximation. An interesting problem is the derivation of formulas (3.33)–(3.35) based on the functional-analytic approach to thermohaline turbulence initiated by the author [2]. Such work is of considerable interest from a theoretical point of view but in the light of the present knowledge, it is an extremely difficult problem.

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